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On some mathematical aspects of the finite element approximation of Darcy's problem

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Abstract. *In this document we summarize some of our recent work on the finite element approximation of Darcy's problem. We start with some general considerations about the problem setting, emphasizing the importance of the functional framework. The Brinkman problem, including viscosity, is analyzed next, in a format that permits to consider the limit cases of the pure Darcy and Stokes problems. Then we move to the finite element approximation, which is a stabilized finite element method that allows one to use arbitrary interpolations for the variables. Improved results for Darcy's problem obtained using duality arguments are then presented. As a last topic, we study the imposition of essential boundary conditions on non-matching methods used what we call the linked-Lagrange multiplier method.*

1 Darcy's problem: primal and dual forms

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) the (bounded) domain where the problem is posed, with boundary $\Gamma = \partial\Omega$. Darcy's problem consists of finding $p : \Omega \longrightarrow \mathbb{R}$ and

$\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ such that

$$-\frac{1}{k}\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = g \quad \text{in } \Omega, \tag{2}$$

where $\boldsymbol{f}$ and g are given functions and $k > 0$ is a physical parameter, here assumed constant. The essential boundary conditions depend on the functional setting of the problem described next. From now on, vector or tensor fields will be represented by boldface characters.

The number of components of the unknown is $n = d+1$, which we arrange as $[\boldsymbol{u}, p]$. The differential operator is $\mathcal{L}([\boldsymbol{u}, p]) = [-\frac{1}{k}\boldsymbol{u} - \nabla p, \nabla \cdot \boldsymbol{u}]$. If we test it against $[\boldsymbol{v}, q]$, we formally obtain a weak form of the problem, which may change depending on the term we integrate by parts. In general, it can be written as

$$\langle \mathcal{L}[\boldsymbol{u}, p], [\boldsymbol{v}, q] \rangle = B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) - \langle \mathcal{F}_n[\boldsymbol{u}, p], \mathcal{D}[\boldsymbol{v}, q] \rangle_\Gamma, \tag{3}$$

where $\mathcal{F}_n$ is the normal trace of a certain operator $\mathcal{F}$. Considering smooth enough functions, we have

$$\begin{aligned} &\langle \mathcal{L}([\boldsymbol{u}, p], [\boldsymbol{v}, q] \rangle \\ &= \begin{cases} -\frac{1}{k}(\boldsymbol{u}, \boldsymbol{v}) - (\nabla p, \boldsymbol{v}) - (\nabla q, \boldsymbol{u}) + \langle \boldsymbol{n} \cdot \boldsymbol{u}, q \rangle_\Gamma, & \text{(P)} \\ -\frac{1}{k}(\boldsymbol{u}, \boldsymbol{v}) + (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}) - \langle \boldsymbol{n} \cdot \boldsymbol{v}, p \rangle_\Gamma. & \text{(D)} \end{cases} \end{aligned}$$

Let Q and V the spaces where p and $\boldsymbol{u}$ are defined, respectively. These depend on whether expression (P) or expression (D) is used. The first leads to the primal form of Darcy's problem and the second to the dual form. The spaces in play, the bilinear form B, the trace space Λ and the functional F whose optimization leads to the Euler-Lagrange equations (1)-(2) are all given in Table 1. The notation used in this table is standard; in particular, $H(\mathrm{div}; \Omega)$ is the spaces of vector fields in $L^2(\Omega)^d$ with divergence in $L^2(\Omega)$. The right-hand-side of the variational form of the problem is $L([\boldsymbol{v}, q]) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \langle g, q \rangle$. Here and below, $\langle \cdot, \cdot \rangle$ stands for the integral over Ω of the product of two functions, whereas if the integral is over another domain ω a subscript is introduced; if the functions belong to $L^2(\Omega)$, the symbol used is $(\cdot, \cdot)$.

In view of the expression of the boundary operator $\mathcal{D}$, the essential boundary conditions that we shall consider are

$$\begin{aligned} &p = \bar{p} \quad \text{on } \Gamma && \text{for the primal form,} \\ &\boldsymbol{n} \cdot \boldsymbol{u} = \bar{u}_n \quad \text{on } \Gamma && \text{for the dual form.} \end{aligned}$$

	Primal form	Dual form
Q	$H^1(\Omega)$	$L^2(\Omega)$
V	$L^2(\Omega)^d$	$H(\mathrm{div};\Omega)$
$B([\boldsymbol{u},p],[\boldsymbol{v},q])$	$-\frac{1}{k}(\boldsymbol{u},\boldsymbol{v})$ $-(\nabla p,\boldsymbol{v})-(\nabla q,\boldsymbol{u})$	$-\frac{1}{k}(\boldsymbol{u},\boldsymbol{v})$ $+(p,\nabla\cdot\boldsymbol{v})+(q,\nabla\cdot\boldsymbol{u})$
$\mathcal{F}[\boldsymbol{u},p]$	$-\boldsymbol{u}$	p
$\mathcal{D}[\boldsymbol{u},p]$	p	$\boldsymbol{n}\cdot\boldsymbol{u}$
Λ	$H^{1/2}(\Gamma)$	$H^{-1/2}(\Gamma)$
$F[\boldsymbol{u},p]$	$-\frac{1}{2k}\lVert\boldsymbol{u}\rVert^2-(\nabla p,\boldsymbol{u})$ $-\langle\boldsymbol{f},\boldsymbol{u}\rangle-\langle g,p\rangle$	$-\frac{1}{2k}\lVert\boldsymbol{u}\rVert^2+(p,\nabla\cdot\boldsymbol{u})$ $-\langle\boldsymbol{f},\boldsymbol{u}\rangle-\langle g,p\rangle$

Table 1: Primal and dual forms of Darcy's problem

2 Brinkman's problem: a functional framework encompassing limit cases

Let us consider now the Stokes-Darcy (or Brinkman) problem. Using the terminology of fluid mechanics, it consists of finding a velocity $\boldsymbol{u}:\Omega\longrightarrow\mathbb{R}^d$ and a pressure $p:\Omega\longrightarrow\mathbb{R}$ such that

$$-\nu\Delta\boldsymbol{u}+\frac{1}{k}\boldsymbol{u}+\nabla p=-\boldsymbol{f}, \tag{4}$$

$$\nabla\cdot\boldsymbol{u}=g. \tag{5}$$

For simplicity, as boundary conditions we will consider $\boldsymbol{u}=\boldsymbol{0}$ if $\nu>0$ and $\boldsymbol{n}\cdot\boldsymbol{u}=0$ if $\nu=0$.

Our objective is to develop a functional framework for the problem *well behaved* when $\nu\to 0$ (zero viscosity) and when $\frac{1}{k}\to 0$ (infinite permeability). In the following section we will propose finite element methods to approximate the problem with optimal stability and convergence properties using *arbitrary* conforming approximations of velocity and pressure, without the difficulty inherent to inf-sup stable elements for both the Stokes and the Darcy problems. The results to be presented can be found in [1].

The variational formulation of (4)-(5) consists in finding a velocity-pressure pair $[\boldsymbol{u},p]$ such that

$$B([\boldsymbol{u},p],[\boldsymbol{v},q])=L([\boldsymbol{v},q]),$$

for all test functions $[\boldsymbol{v},q]$, where the bilinear form B and the linear form L are now defined by

$$B([\boldsymbol{u},p],[\boldsymbol{v},q])=\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+\frac{1}{k}(\boldsymbol{u},\boldsymbol{v})-(p,\nabla\cdot\boldsymbol{v})+(q,\nabla\cdot\boldsymbol{u}),$$
$$L([\boldsymbol{v},q])=-\langle\boldsymbol{f},\boldsymbol{v}\rangle+\langle g,q\rangle.$$

As shown in the previous section, if $\nu = 0$, that is, for the Darcy problem, the problem can be posed in the primal and the dual forms. The latter corresponds to the singular limit $\nu \to 0$, where it is natural to require that $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$, $g \in L^2(\Omega)$. The primal form is just a mixed formulation of Poisson's problem and the data need to be $\boldsymbol{f} \in L^2(\Omega)^d$, $g \in H^{-1}(\Omega)$.

Let us introduce the operator

$$\mathcal{L}\boldsymbol{u} := -\nu\Delta\boldsymbol{u} + \frac{1}{k}\boldsymbol{u},$$

and the associated graph norm

$$\|\boldsymbol{u}\|_{\mathcal{L}}^2 := \nu\|\nabla\boldsymbol{u}\|^2 + \frac{1}{k}\|\boldsymbol{u}\|^2.$$

Let $V_{\mathcal{L}}$ be obtained as the closure of $C_0^\infty(\Omega)^d$ with respect to this norm. Its dual space $V_{\mathcal{L}}'$ is endowed with the norm

$$\|\boldsymbol{u}\|_{\mathcal{L}'} := \sup_{\boldsymbol{v}\in V_{\mathcal{L}}} \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\|\boldsymbol{v}\|_{\mathcal{L}}}.$$

Obviously, $V_{\mathcal{L}} = H_0^1(\Omega)^d$, $V_{\mathcal{L}}' = H^{-1}(\Omega)^d$ if $\nu > 0$ and $V_{\mathcal{L}} = V_{\mathcal{L}}' = L^2(\Omega)^d$ if $\nu = 0$.

A key ingredient is the introduction of *a characteristic length scale* L_0, which will play a key role in the Darcy problem. The reason is the need to control both $\boldsymbol{u}$ and $\nabla\cdot\boldsymbol{u}$ to obtain stability in $H(\mathrm{div}, \Omega)$.

Let V be the closure of $C_0^\infty(\Omega)^d$ with respect to the norm given by $\|\boldsymbol{v}\|_{\mathcal{L}} + k^{-1/2}L_0\|\nabla\cdot\boldsymbol{v}\|$ and Q the closure of $C^\infty(\Omega)/\mathbb{R}$ with respect to the norm $(\nu+k^{-1}L_0^2)^{-1/2}\|q\|+\|\nabla q\|_{\mathcal{L}'}$. The pair $V\times Q$ reduces to $H_0^1(\Omega)^d\times L^2(\Omega)/\mathbb{R}$ when $\nu > 0$ and to $H_0(\mathrm{div},\Omega)\times H^1(\Omega)/\mathbb{R}$ when $\nu = 0$. On $V\times Q$ we define

$$|||[\boldsymbol{v}, q]|||^2 := \|\boldsymbol{v}\|_{\mathcal{L}}^2 + \frac{1}{k}L_0^2\|\nabla\cdot\boldsymbol{v}\|^2 + \frac{1}{\nu + k^{-1}L_0^2}\|q\|^2 + \|\nabla q\|_{\mathcal{L}'}^2, \quad (6)$$

which is the finest norm in which the problem is well posed.

Theorem 1 [Stability of the continuous problem] *There exists a constant C such that for all $[\boldsymbol{u}, p] \in V\times Q$ there exists $[\boldsymbol{v}, q] \in V_{\mathcal{L}}\times L^2(\Omega)$ for which*

$$B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) \geq C|||[\boldsymbol{u}, p]|||\;\|[\boldsymbol{v}, q]\|_{V_{\mathcal{L}}\times L^2(\Omega)},$$

i.e., the problem is well-posed in the norm (6).

The working norm is optimal. Observe that

$$|||[\boldsymbol{u}, p]|||^2 = \nu\|\nabla\boldsymbol{u}\|^2 + \frac{1}{\nu}\|p\|^2 + \frac{1}{\nu}\|\nabla p\|_{-1}^2 \qquad \text{when } \tfrac{1}{k} = 0,$$

$$|||[\boldsymbol{u}, p]|||^2 = \frac{1}{k}\|\boldsymbol{u}\|^2 + \frac{1}{k}L_0^2\|\nabla\cdot\boldsymbol{u}\|^2 + \frac{k}{L_0^2}\|p\|^2 + k\|\nabla p\|^2 \quad \text{when } \nu = 0.$$

3 Brinkman's problem: stabilized finite element approximation

Once the functional framework for the Brinkman problem has been presented, let us introduce the finite element approximation we propose using stabilized finite element methods. What follows is a summary of [1, 3].

The key ingredients of the method we will present are a two scale decomposition of the velocity and the pressure into finite element components and *subscales* within the variational multiscale framework (VMS), a proper *scaling* of the problem, which requires the introduction of a *length scale* and a closed form expression for the subscales based on an approximate Fourier analysis of the problem (not elaborated here). As a result, our proposal consists of *two stabilized finite element methods*, with similar stability and convergence properties, namely, optimal stability and optimal convergence in the appropriate functional setting of the problem.

Let V_h and Q_h be the finite element spaces to approximate the velocity and the pressure, constructed from a finite element partition $\{K\}$. The methods to be analyzed can be written as follows: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_s([\boldsymbol{v}_h, q_h]), \tag{7}$$

for all $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$.

The first method we consider is the *Algebraic subgrid scale (ASGS) method*, in which the forms B_s and L_s in (7) are given by:

$$\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\
&+\tau_p \sum_K \langle \nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h\rangle_K \\
&+\tau_u \sum_K \left\langle -\nu\Delta\boldsymbol{u}_h + k^{-1}\boldsymbol{u}_h + \nabla p_h, \nu\Delta\boldsymbol{v}_h - k^{-1}\boldsymbol{v}_h + \nabla q_h\right\rangle_K \\
&+\tau_f \sum_E \langle [\![\boldsymbol{n}p_h - \nu\partial_n\boldsymbol{u}_h]\!], [\![\boldsymbol{n}q_h + \nu\partial_n\boldsymbol{v}_h]\!]\rangle_E,
\end{aligned}$$

where $[\![\cdot]\!]$ is the jump over the edges E and

$$\begin{aligned}
L_s([\boldsymbol{v}_h, q_h]) &= L([\boldsymbol{v}_h, q_h]) + \tau_p \sum_K \langle g, \nabla\cdot\boldsymbol{v}_h\rangle_K \\
&+\tau_u \sum_K \left\langle -\boldsymbol{f}, \nu\Delta\boldsymbol{v}_h - k^{-1}\boldsymbol{v}_h + \nabla q_h\right\rangle_K.
\end{aligned}$$

The formulation depends on the stabilization parameters τ_p, τ_u and τ_f, that we compute as

$$\tau_p = c_1\nu + c_2^p\frac{\ell_p^2}{k}, \quad \tau_u = \left(c_1\nu + c_2^u\frac{\ell_u^2}{k}\right)^{-1} h^2, \quad \tau_f = \frac{\tau_u}{h}, \tag{8}$$

with c_1, c_2^p and c_2^u algorithmic constants.

The length scales ℓ_u and ℓ_p, which can be either taken as L_0, h or $(L_0 h)^{1/2}$, appear when introducing scaling coefficients μ_u and μ_p such that $\mu_u|\boldsymbol{f}|^2 + \mu_p|g|^2$ is dimensionally consistent. Using the approximate Fourier analysis, the stabilization parameters are found, now depending on μ_u and μ_p. In turn, these scaling coefficients depend on a length scale of the problem that may be taken as L_0 or h.

The second method we consider is the *Orthogonal subscale stabilization (OSS) method*. In this case, the bilinear form B_s and the linear form L_s in (7) for the OSS method are given by

$$\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\
&\quad + \tau_p \sum_K \Big\langle P^\perp(\nabla\cdot\boldsymbol{u}_h), P^\perp(\nabla\cdot\boldsymbol{v}_h)\Big\rangle_K \\
&\quad + \tau_u \sum_K \Big\langle P^\perp(-\nu\boldsymbol{v}_h + \nabla p_h), P^\perp(\nu\Delta\boldsymbol{v}_h + \nabla q_h)\Big\rangle_K \\
&\quad + \tau_f \sum_E \langle [\![\boldsymbol{n}p_h - \nu\partial_n\boldsymbol{u}_h]\!], [\![\boldsymbol{n}q_h + \nu\partial_n\boldsymbol{v}_h]\!]\rangle_E, \\
L_s([\boldsymbol{v}_h, q_h]) &= L([\boldsymbol{v}_h, q_h]).
\end{aligned}$$

The stabilization parameters are the same as for the ASGS method.

Both for the ASGS and the OSS methods, let us define the mesh dependent norm

$$\begin{aligned}
|\!|\!|[\boldsymbol{v}_h, q_h]|\!|\!|_h^2 &= \|\boldsymbol{v}_h\|_{\mathcal{L}}^2 + \frac{1}{k}\ell_p^2\|\nabla\cdot\boldsymbol{v}_h\|^2 + \frac{1}{\nu + k^{-1}L_0^2}\|q_h\|^2 \\
&\quad + \frac{h^2}{\nu + k^{-1}\ell_u^2}\sum_K \|\nabla q_h\|_K^2 + \frac{h}{\nu + k^{-1}\ell_u^2}\sum_E \|[\![\boldsymbol{n}q_h]\!]\|_E^2. \qquad (9)
\end{aligned}$$

If $\varepsilon_i(v)$ is the interpolation error of function v in the norm of $H^i(\Omega)$, let us define

$$\begin{aligned}
\mathcal{E}(h)^2 &= (\nu + k^{-1}\ell_p^2)(h^{-2}\varepsilon_0^2(\boldsymbol{u}) + \varepsilon_1^2(\boldsymbol{u})) + k^{-1}\varepsilon_0^2(\boldsymbol{u}) \\
&\quad + \frac{h^2}{\nu + k^{-1}\ell_u^2}(h^{-2}\varepsilon_0^2(p) + \varepsilon_1^2(p)). \qquad (10)
\end{aligned}$$

Theorem 2 [Stability of the stabilized formulations] *Suppose that the constants* c_1 *and* c_2^u *are large enough. Then, there exists a constant* C *such that for all* $[\boldsymbol{u}_h, p_h]$ *there exists* $[\boldsymbol{v}_h, q_h]$ *such that*

$$B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \geq C|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_h |\!|\!|[\boldsymbol{v}_h, q_h]|\!|\!|_h,$$

i.e., the discrete problem is stable in the norm (9).

Let us compare the working norms of the continuous and the discrete problems, for simplicity in the case of continuous pressure interpolations:

$$
\begin{aligned}
\|\![\boldsymbol{v}, q]\|\!|^2 &= \|\boldsymbol{v}\|_{\mathcal{L}}^2 + \frac{1}{\nu + k^{-1}L_0^2}\|q\|^2 + \frac{1}{k}L_0^2\|\nabla\cdot\boldsymbol{v}\|^2 + \|\nabla q\|_{\mathcal{L}'}^2, \\
\|\![\boldsymbol{v}_h, q_h]\|\!|_h^2 &= \|\boldsymbol{v}_h\|_{\mathcal{L}}^2 + \frac{1}{\nu + k^{-1}L_0^2}\|q_h\|^2 \\
&\quad + \frac{1}{k}\ell_p^2\|\nabla\cdot\boldsymbol{v}_h\|^2 + \frac{h^2}{\nu + k^{-1}\ell_u^2}\|\nabla q_h\|^2.
\end{aligned}
$$

If $\lesssim$ stands for $\leq$ up to constants, we have:

Theorem 3 [Convergence of the stabilized formulations] *Let $[\boldsymbol{u}, p]$ be the solution of the continuous problem and $[\boldsymbol{u}_h, p_h]$ the solution of the discrete one. Suppose that $\ell_p \geq \ell_u$ and the assumptions of the previous theorem hold. Then*

$$\|\![\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]\|\!|_h \lesssim \mathcal{E}(h),$$

i.e., the finite element solution converges in the norm (9) with error function (10).

4 Optimal estimates for Darcy's problem using stabilized finite element methods

Let us give now error estimates for the Darcy problem. The previous results also hold in the case $\nu = 0$, so that we already have results in natural norms. However, we will show that it is possible to improve them using duality arguments. Although we shall not treat them, the same results can be obtained using discontinuous approximations for both the pressure and the velocity (see [2]).

Let $\boldsymbol{e}_u$ and e_p be the finite element errors in velocity and pressure, respectively. The estimates obtained before in the case $\nu = 0$ can be written as

$$
\begin{aligned}
&\frac{1}{k}\|\boldsymbol{e}_u\|^2 + \frac{1}{k}\ell_p^2\|\nabla\cdot\boldsymbol{e}_u\|^2 + \frac{k}{L_0^2}\|e_p\|^2 \\
&\quad + \frac{kh^2}{\ell_u^2}\sum_K \|\nabla e_p\|_K^2 + \frac{kh}{\ell_u^2}\sum_E \|[\![\boldsymbol{n} e_p]\!]\|_E^2 \\
&\quad \lesssim \frac{1}{k}\ell_p^2\varepsilon_1^2(\boldsymbol{u}) + \frac{1}{k}\varepsilon_0^2(\boldsymbol{u}) + \frac{k}{\ell_u^2}\varepsilon_0^2(p) \\
&\quad \lesssim \frac{1}{k}\ell_p^2 h^{2k}\|\boldsymbol{u}\|_{k+1}^2 + \frac{1}{k}h^{2k+2}\|\boldsymbol{u}\|_{k+1}^2 + \frac{k}{\ell_u^2}h^{2l+2}\|p\|_{l+1}^2,
\end{aligned}
$$

where a subscript m in $\|\cdot\|$ denotes the H^m norm. In order to obtain improved error estimates in $L^2(\Omega)$ for $\boldsymbol{e}_u$ and e_p we need to assume that the adjoint problem satisfies the usual elliptic regularity assumption.

The bilinear form of the problem using the ASGS formulation can then be written as

$$
\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + \frac{\ell_p^2}{k} \sum_K \langle \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h \rangle_K \\
&+ \frac{h^2}{k\ell_u^2} \sum_K \langle \boldsymbol{u}_h + k\nabla p_h, -\boldsymbol{v}_h + k\nabla q_h \rangle_K + \frac{kh}{\ell_u^2} \sum_E \langle [\![\boldsymbol{n} p_h]\!], [\![\boldsymbol{n} q_h]\!] \rangle_E,
\end{aligned}
$$

and for the OSS method

$$
\begin{aligned}
&B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\
&\quad + \frac{\ell^2}{k} \sum_K \left\langle \nabla \cdot \boldsymbol{v}_h, P^\perp(\nabla \cdot \boldsymbol{u}_h) \right\rangle_K \\
&\quad + \frac{kh^2}{\ell^2} \sum_K \left\langle \nabla q_h, P^\perp(\nabla p_h) \right\rangle_K + \frac{kh}{\ell^2} \sum_E \langle [\![\boldsymbol{n} p_h]\!], [\![\boldsymbol{n} q_h]\!] \rangle_E,
\end{aligned}
$$

where $\ell_u = \ell_p = \ell$ has been assumed.

Theorem 4 [Optimal convergence for Darcy's problem] *There holds*

$$
\begin{aligned}
\|\boldsymbol{e}_u\|^2 &\lesssim \Big(h^2 + \frac{\ell_p^4}{L_0^2} + h^2 \frac{\ell_p^4}{\ell_u^4} \Big) \|\nabla \cdot \boldsymbol{e}_u\|^2 + k^2 \Big(\frac{h^4}{\ell_u^4} + \frac{h^2}{L_0^2} \Big) \sum_K \|\nabla e_p\|_K^2, \\
\|e_p\|^2 &\lesssim \frac{1}{k^2} \ell_p^4 \|\nabla \cdot \boldsymbol{e}_u\|^2 + h^2 \sum_K \|\nabla e_p\|_K^2.
\end{aligned}
$$

The accuracy of the velocity and the pressure implied by this result is shown in Table 2 in terms of the length scale chosen. The important point is that the choice of the length scales determines the functional setting in which we converge, either that of the primal form or of the dual form of Darcy's problem.

5 Weak imposition of essential boundary conditions

Let us describe now a methodology that we have proposed to impose in a weak way essential boundary conditions for elliptic problems and, in particular, for the Darcy problem. The formulation is applicable if the finite element mesh covers a domain Ω_h which does not match Ω, i.e., Γ is interior to Ω_h.

Method $\ell_p, \ell_u =$	h, h	${L_0}^{1/2}h^{1/2}, {L_0}^{1/2}h^{1/2}$	L_0, L_0
$\Vert \boldsymbol{e}_u \Vert$ Original	$h^{k+1} + h^l$ Suboptimal	$h^{k+1/2} + h^{l+1/2}$ Quasi-optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert \boldsymbol{e}_u \Vert$ Via duality	$h^{k+1} + h^l$ Suboptimal	$h^{k+1} + h^{l+1}$ Optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert e_p \Vert$ Original	$h^{k+1} + h^l$ Suboptimal	$h^{k+1/2} + h^{l+1/2}$ Quasi-optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert e_p \Vert$ Via duality	$h^{k+2} + h^{l+1}$ Optimal	$h^{k+1} + h^{l+1}$ Optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert \nabla \cdot \boldsymbol{e}_u \Vert$	$h^k + h^{l-1}$ Suboptimal	$h^k + h^l$ Optimal	$h^k + h^{l+1}$ Optimal
$\Vert \nabla e_p \Vert$	$h^{k+1} + h^l$ Optimal	$h^k + h^l$ Optimal	$h^{k-1} + h^l$ Suboptimal
$k,\ l$ Optimal	$k+1=l$	$k=l$	$k=l+1$

Table 2: Accuracy of the stabilized formulations for Darcy's problem

The method has been proposed in a preliminary version in [5], it was generalized in [6] and the full numerical analysis can be found in [4]. The reader is referred to these articles and references therein for details.

The formulation we proposed is termed *Linked Lagrange Multiplier* (LLM) method. The idea can be explained starting from the classical Lagrange Multiplier method to impose boundary conditions. Instead of considering the multiplier as a variable defined on the boundary, we consider it as the trace of a certain field defined on the whole domain, and impose that this field be equal to a certain operator applied to the unknowns of the problem in a least-squares sense. In the particular case of Darcy's problem, the method consists in the optimization of the functional

$$\begin{aligned}&\hat{G}([\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h])\\ &= \begin{cases} F([\boldsymbol{u}_h, p_h]) - \langle \sigma_{n,h}, p_h - \bar{p} \rangle_\Gamma - \frac{1}{2N_0 k} \Vert \boldsymbol{\sigma}_h + \boldsymbol{u}_h \Vert^2, & \text{(P)} \\ F([\boldsymbol{u}_h, p_h]) - \langle \sigma_h, \boldsymbol{n} \cdot \boldsymbol{u}_h - \bar{u}_n \rangle_\Gamma - \frac{k}{2N_0 L_0^2} \Vert \sigma_h - p_h \Vert^2, & \text{(D)} \end{cases}\end{aligned}$$

over $V_h \times Q_h \times \Sigma_h$, corresponding to the primal (P) and the dual (D) forms of the problem. Note that in the first case $\boldsymbol{\sigma}_h$ is a vector field, with the physical meaning of a flux, and $\sigma_{n,h}$ is its normal trace on Γ, whereas in the second case σ_h is a scalar, with the same physical meaning as the primal variable. The factors k and L_0^2 in the least-squares terms have been introduced to leave only one dimensionless parameter N_0 in the formulation. The space for the new variable, Σ_h, can be chosen to be made of discontinuous functions, thus allowing the condensation of this new unknown at the element level.

Let us start with the primal form. It reads: find $[\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h] \in V_h \times Q_h \times \Sigma_h$ such that

$$-\frac{1}{k}(\boldsymbol{u}_h, \boldsymbol{v}_h) - (\nabla p_h, \boldsymbol{v}_h) - \frac{1}{N_0 k}(\boldsymbol{v}_h, \boldsymbol{\sigma}_h + \boldsymbol{u}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle, \tag{11}$$

$$-(\nabla q_h, \boldsymbol{u}_h) - \langle \sigma_{n,h}, q_h \rangle_\Gamma = \langle g, q_h \rangle, \tag{12}$$

$$-\langle \tau_{n,h}, p_h \rangle_\Gamma - \frac{1}{N_0 k}(\boldsymbol{\tau}_h, \boldsymbol{\sigma}_h + \boldsymbol{u}_h) = -\langle \tau_{n,h}, \bar{p} \rangle_\Gamma, \tag{13}$$

for all $\boldsymbol{v}_h \in V_h$, $q_h \in Q_h$ and $\boldsymbol{\tau}_h \in \Sigma_h$. The bilinear form associated to problem (11)-(13) is given by

$$\begin{aligned} &B_{\mathrm{DP}}([\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h], [\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h]) \\ &\quad = -\frac{1}{k}(\boldsymbol{u}_h, \boldsymbol{v}_h) - (\nabla p_h, \boldsymbol{v}_h) - (\nabla q_h, \boldsymbol{u}_h) \\ &\quad - \langle \sigma_{n,h}, q_h \rangle_\Gamma - \langle \tau_{n,h}, u_h \rangle_\Gamma - \frac{1}{N_0 k}(\boldsymbol{\tau}_h + \boldsymbol{v}_h, \boldsymbol{\sigma}_h + \boldsymbol{u}_h). \end{aligned} \tag{14}$$

We can prove that [4]:

Theorem 5 [Stability and convergence for the primal problem] *Suppose that Σ_h is made of discontinuous functions, that $N_0 > 1$ and that the pair $Q_h \times V_h$ is inf-sup stable. Then, the bilinear form in (14) is stable in the norm*

$$|||[\boldsymbol{u}, p, \boldsymbol{\sigma}]|||_{\mathrm{DP}}^2 := \frac{1}{k}\|\boldsymbol{u}\|^2 + k\|\nabla p\|^2 + \frac{k}{h}\|p\|_\Gamma^2 + \frac{1}{k}\|\boldsymbol{\sigma}\|^2. \tag{15}$$

Moreover, if the solution of the continuous problem $\mathbf{u} = [\boldsymbol{u}, p, -\boldsymbol{u}]$ is such that $\boldsymbol{n} \cdot \boldsymbol{u}$ is bounded in $L^2(\Gamma)$, the solution $\mathbf{u}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h]$ of problem (11)-(13) converges in the norm (15) with the error function

$$\begin{aligned} \mathcal{E}_{\mathrm{DP}}(\mathbf{u}, h) = &\, k^{-1/2}\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\| + k^{1/2}\|\nabla p - \nabla \tilde{p}_h\| \\ &+ k^{-1/2}\|\boldsymbol{u} + \tilde{\boldsymbol{\sigma}}_h\| + k^{-1/2}h^{1/2}\|\boldsymbol{n} \cdot \boldsymbol{u} + \boldsymbol{n} \cdot \tilde{\boldsymbol{\sigma}}_h\|_\Gamma \\ &+ k^{1/2}h^{-1/2}\|p - \tilde{p}_h\|_\Gamma \end{aligned} \tag{16}$$

for any $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h] \in V_h \times Q_h \times \Sigma_h$.

It is seen from (16) that the error estimate obtained is optimal. In particular, if Q_h is constructed with elements of order p_Q, V_h with elements of order p_V and Σ_h with (discontinuous) elements of order p_Σ, the error in the norm (15) is of order $\min\{p_Q, p_V + 1, p_\Sigma + 1\}$.

Let us move to the dual form of Darcy's problem. The method we propose reads: find $[\boldsymbol{u}_h, p_h, \sigma_h] \in V_h \times Q_h \times \Sigma_h$ such that

$$-\frac{1}{k}(\boldsymbol{u}_h, \boldsymbol{v}_h) + (p_h, \nabla \cdot \boldsymbol{v}_h) - \langle \sigma_h, \boldsymbol{n} \cdot \boldsymbol{v}_h \rangle_\Gamma = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle, \tag{17}$$

$$(q_h, \nabla \cdot \boldsymbol{u}_h) - \frac{k}{N_0 L_0^2}(q_h, \sigma_h - p_h) = \langle g, q_h \rangle, \tag{18}$$

$$-\langle \tau_h, \boldsymbol{n} \cdot \boldsymbol{u}_h \rangle_\Gamma + \frac{k}{N_0 L_0^2}(\tau_h, \sigma_h - p_h) = -\langle \tau_h, \bar{u}_n \rangle_\Gamma, \tag{19}$$

for all $\boldsymbol{v}_h \in V_h$, $q_h \in Q_h$ and $\tau_h \in \Sigma_h$. This is the LLM for the dual form of Darcy's problem. As for the primal version, the linked Lagrange multiplier σ_h can be condensed at the element level from (19) if its interpolation is discontinuous.

The bilinear form associated to problem (17)-(19) is given by

$$\begin{aligned} &B_{\mathrm{DD}}([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, q_h, \tau_h]) \\ &= -\frac{1}{k}(\boldsymbol{u}_h, \boldsymbol{v}_h) + (p_h, \nabla \cdot \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) \\ &- \langle \tau_h, \boldsymbol{n} \cdot \boldsymbol{u}_h \rangle_\Gamma - \langle \sigma_h, \boldsymbol{n} \cdot \boldsymbol{v}_h \rangle_\Gamma + \frac{k}{N_0 L_0^2}(\tau_h - q_h, \sigma_h - p_h). \end{aligned} \tag{20}$$

The stability and convergence result for the problem we are considering is the following [4]:

Theorem 6 [Stability and convergence for the dual problem] *Suppose that* Σ_h *is made of discontinuous functions of order* $p_\Sigma \geq p_V$*, that* $N_0 > 1$ *and that the inf-sup conditions between* Q_h *and* V_h *hold. Then, the bilinear form in (20) is stable in the norm*

$$\begin{aligned} |||[\boldsymbol{u}, p, \sigma]|||_{\mathrm{DD}}^2 := \frac{1}{k}\|\boldsymbol{u}\|^2 + \frac{L_0^2}{k}\|\nabla \cdot \boldsymbol{u}\|^2 + \frac{L_0^2}{kh}\|\boldsymbol{n} \cdot \boldsymbol{u}\|_\Gamma^2 \\ + \frac{k}{L_0^2}\|p\|^2 + \frac{k}{L_0^2}\|\sigma\|^2. \end{aligned} \tag{21}$$

Moreover, if the solution of the continuous problem is $\mathbf{u} = [\boldsymbol{u}, p, p]$*, the solution* $\mathbf{u}_h = [\boldsymbol{u}_h, p_h, \sigma_h]$ *of problem (17)-(19) converges in the norm (21) with the error function*

$$\begin{aligned} \mathcal{E}_{\mathrm{DD}}(\boldsymbol{u}, h) &= k^{-1/2}\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\| + k^{-1/2}L_0\|\nabla \cdot \boldsymbol{u} - \nabla \cdot \tilde{\boldsymbol{u}}_h\| \\ &+ k^{-1/2}h^{-1/2}L_0\|\boldsymbol{n} \cdot \boldsymbol{u} - \boldsymbol{n} \cdot \tilde{\boldsymbol{u}}_h\|_\Gamma + k^{1/2}L_0^{-1}\|p - \tilde{p}_h\| \\ &+ k^{1/2}L_0^{-1}\|p - \tilde{\sigma}_h\| + k^{1/2}h^{1/2}L_0^{-1}\|p - \tilde{p}_h\|_\Gamma \end{aligned} \tag{22}$$

for any $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h] \in V_h \times Q_h \times \Sigma_h$.

As for the primal problem, it is observed from (22) that the error estimate obtained is optimal.

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